Homework 6

This assignment is due by **11:59pm on Friday 10/25** on gradescope. You should start a new page for each problem (if a problem has multiple parts, the parts can be completed on the same page). After completing the assignment, save your solutions as a single pdf and upload it to gradescope. Be sure to **assign your pages** on the gradescope upload page; failure to do so will result in a penalty.

You may collaborate with your peers on homework assignments and to discuss points of confusion with the instructor and/or TAs. However, you must write your own solutions independently and you may not look at or copy any one else's solutions. You must show all of your work to receive full credit (with the exception of row operations).

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with associated matrix $Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.
   (a) Let $\theta = \frac{\pi}{4}$. Evaluate T at the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Sketch the original vectors and images in two separate $xy-$planes. What does T do to these vectors, geometrically, in terms of θ?
   (b) Show that Q_θ is orthogonal (for all θ) using the definition of orthogonal matrices.
2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with associated matrix $R_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.
   (a) Let $\theta = \frac{\pi}{4}$. Evaluate T at the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Sketch the original vectors and images in two separate $xy-$planes. What does T do to these vectors, geometrically, in terms of θ?
   (b) Show that R_θ is orthogonal (for all θ) using the definition of orthogonal matrices.
3. The matrices Q_θ and R_θ defined in the above problems are called rotation and reflection matrices, respectively. Show that any 2×2 orthogonal matrix is either a rotation or reflection matrix.
   Hint: Use the fact that any unit vector in $\mathbb{R}^2$ is of the form $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ along with various trig identities.
4. Let $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be an orthogonal basis of a subspace W of $\mathbb{R}^m$. Show that
$$\text{proj}_W \mathbf{b} = \text{proj}_{\mathbf{c}_1} \mathbf{b} + \cdots + \text{proj}_{\mathbf{c}_n} \mathbf{b}.$$
   Hint: Multiply out the formula $\text{proj}_W \mathbf{b} = A(A^T A)^{-1} A^T \mathbf{b}$.

5. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

   (a) Use Gram-Schmidt to find an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ of W.

   (b) Use the orthonormal basis found in part (a) to find the projection of $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ onto W by using the formula in problem 4.

   (c) Use part (a) to find a QR factorization of $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$.

   (d) Use the QR factorization to find the least squares solution to $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2/\sqrt{3} \end{bmatrix}$. Provide an exact answer (no rounding).

6. Let Q be an orthogonal matrix. Show that $\det Q = \pm 1$ in two ways.
   (a) Algebraically, using properties of determinants and orthogonal matrices.
   (b) Geometrically, using intuition.

7. Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix}$. Find $\det A$ using row reduction and the properties and discussed in class. Be sure to explain your steps.

8. Recall that the permutation matrix P_{ij} is the matrix obtained by swapping the two rows of I. On Homework 3 you showed that $P_{ij}^{-1} = P_{ij}$. Let P be a product of r permutation matrices (P is also called a permutation matrix).
   (a) Use determinant axioms/properties discussed in class to show that $\det P = (-1)^r$.
   (b) Show that $P^{-1} = P^T$
   (c) Use the above parts to show that $\det P = \det P^T$.
   [This fact is needed to prove the general property $\det A = \det A^T$ (see lecture notes); so you cannot use the general property to complete this part].